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SCATTERING AND NONSCATTERING OBSTACLES*

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For Joseph B. Keller, friend, colleague and teacher

Abstract. Two problems of Helmholtz's equation for a wave incident on an obstacle are considered. For the first, the scattering problem, the obstacle's response satisfies Sommerfeld's outgoing wave radiation condition, and the net radiative response is positive; for the second, the response satisfies a standing wave condition (an appropriate combination of outgoing and incoming waves) such that the net radiative response is zero. The essential features of the solutions are exhibited in terms of amplitude functions g (the usual scattering amplitude) and g', and the interrelation of the functions are stressed in the derivation of integral equations $g[g']$ (introduced earlier in multiple scattering contexts). The scattering amplitude g is always complex, but the simpler function g' is shown to be imaginary for nonabsorbing obstacles having inversion symmetry. Long-wavelength approximations for g' may be obtained from potential theory and perturbation procedures, and corresponding approximations for g then follow from $g[g']$.

Introduction. We consider two problems for Helmholtz's equations corresponding to a wave incident on an obstacle. The problems differ in the obstacle's response via radiation: for the first, the scattering problem (an outward radiating obstacle), the net radiative output is greater than zero; for the second the net radiative output is zero. For brevity, the obstacle in the second problem is labeled radiationless or nonscattering, but its contribution is in the form of a standing wave so the labels refer to net energy flow via radiation. Section 1 defines the problems, and delineates analytical and physical differences.

The solutions of the problems are characterized by amplitude functions g (the usual scattering amplitude) and g'. Section 2 sketches derivations of the general theorems for g with emphasis on procedures that can be applied for g'; § 3 derives the analogues for g', and § 4 applies the same procedures to derive integral equations $g[g']$ and their inverses. The function g' is simpler than g. The scattering amplitude g is always complex, but g' is shown to be imaginary for lossless (nonabsorbing) obstacles having inversion symmetry. Long-wavelength approximations for g' may be obtained from potential theory solutions and perturbation procedures, and corresponding approximations for g then follow from $g[g']$.

We considered $g'[g] = g'[\int; g]$ initially in connection with multiple scattering of waves by periodic arrays of obstacles [1]. The multiple scattering amplitude (G) of an obstacle in the array was related to its response (g) in isolation by a functional equation $G = G[S; g]$, where the operator S equaled a discrete infinite sum less the analogous integral. It proved convenient to introduce $g'[\int; g]$ corresponding to g stripped of radiation losses and to express G functionally in terms of g' and a modified operator S'. The form $G[S'; g']$ was particularly suitable to analyze radiation losses appropriate for the array. The function g involves radiative losses over the continuum of real directions, but G involves only the set of discrete directions corresponding to the propagating modes of the array [1]. Working with g' led directly to simple energy conserving approximations for G, and to rapidly converging representations. The required properties of $g' = g'[\int; g]$ were deduced from known properties of g.

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Related work on multiple scattering by pair-correlated distributions of obstacles [2] led to functional equations $G[g]$ involving more complicated operators than S. To reduce these to tractable forms $G[g']$ for nonsymmetrical lossy obstacles required additional properties of g'. The present paper derives these additional properties of g' (as well as those obtained earlier from g) by direct considerations of the radiationless obstacle problem.

The following includes material on some elementary properties of the scattering amplitude g and the associated field [3]–[8]. This material serves to distinguish g and g', and to stress that the nonscattering problem for g' does not correspond to a conventional scattering problem subject to special constraints for decreasing the magnitude of g. The nonscattering problem we consider is a standing wave problem ancillary to the scattering problem; similarly for the associated amplitude g' (labeled earlier as the modified or transformed amplitude) and g.

1. Statement of the problems. For the problems at hand corresponding to a plane wave $\phi(\mathbf{r})\,e^{-i\omega t}$ of angular frequency ω incident on an obstacle, we deal with solutions of Helmholtz's equation [3]–[8]

(1) $$(\nabla^2+k^2)\psi=0, \qquad k=|k|$$

where $\nabla^2=\nabla\cdot\nabla$ is Laplace's operator and k is the propagation parameter. We suppress the time factor $e^{-i\omega t}$, but all waves we consider have implicit period $2\pi/\omega$ in t and phase velocity ω/k; the plane waves have explicit period $2\pi/k$ (the wavelength λ) in the space coordinates.

A plane wave is represented by

(2) $$\phi=e^{i\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{k}=k\hat{\mathbf{k}}, \quad \mathbf{r}=r\hat{\mathbf{r}}$$

with directions of incidence and observation given by unit vectors $\hat{\mathbf{k}}$ and $\hat{\mathbf{r}}$. In three dimensions,

(3) $$\hat{\mathbf{r}}=\hat{\mathbf{r}}(\theta,\varphi)=\hat{\mathbf{z}}\cos\theta+(\hat{\mathbf{x}}\cos\varphi+\hat{\mathbf{y}}\sin\varphi)\sin\theta, \qquad \hat{\mathbf{k}}=\hat{\mathbf{r}}_0=\hat{\mathbf{r}}(\theta_0,\varphi_0),$$

with $0\leqq\theta\leqq\pi$, $0\leqq\varphi<2\pi$; in two dimensions, we take $\varphi=0$ and $0\leqq\theta<2\pi$; in one, $\hat{\mathbf{r}}=\pm\hat{\mathbf{z}}$, and $\theta=0,\ \pi$. We consider three-, two-, and one-dimensional problems in parallel (corresponding to bounded obstacles, and to normal incidence on cylinders and slabs respectively), and use three-dimensional terminology for all cases. If three different forms of a function or parameter arise, they are sequenced in order of decreasing dimensionality. The center of the obstacle's smallest circumscribing sphere (of radius a) is taken as the phase origin ($r=0$).

An obstacle (in general, a discontinuity in the parameters associated with the medium in which ϕ propagates) is specified by its surface $\mathcal{S}$, volume $\mathcal{V}$, size ka relative to wavelength, and by conditions on ψ in $\mathcal{S}$ and $\mathcal{V}$. An obstacle is labeled impenetrable if $\psi=0$ in $\mathcal{V}$ and satisfies on $\mathcal{S}$ conditions such as: either

(4) $$\psi=0$$

or

(5) $$\hat{\mathbf{n}}\cdot\nabla\psi=\partial_n\psi=0,$$

with $\hat{\mathbf{n}}$ as the outward normal into the external volume V, or

(6) $$\partial_n\psi=Z\psi,$$

where Z is constant with $\operatorname{Im} Z\leqq 0$. (These are the usual Dirichlet, Neumann, and Robin exterior boundary value conditions respectively.) Physically, e.g., for small

amplitude acoustics with ψ as excess pressure, the first two correspond to free (pressure release) and rigid surfaces respectively, and the third is a generalization in terms of the impedance Z.

An obstacle is labeled penetrable if ψ satisfies transition conditions on $\mathcal{S}$, such as

$$(7\mathcal{S}) \qquad \psi_E = \psi_I, \qquad \partial_n\psi_E = B\,\partial_n\psi_I,$$

where E and I stand for exterior and interior (a temporary convenience), and ψ_I is a nonsingular solution in $\mathcal{V}$ of

$$(7\mathcal{V}) \qquad (\nabla^2+K^2)\psi = 0, \qquad K = k\eta, \quad \eta^2 = C/B.$$

Here the obstacle's relative parameters C and B are constants with $\operatorname{Im} C \geqq 0$ and $\operatorname{Im} B \leqq 0$, and η is the corresponding complex relative index of refraction. In acoustics, for real values, C is the obstacle's relative compressibility and $1/B$ its relative mass density; complex values account for some thermal loss effects, compressive viscosity, etc. The $\mathcal{S}$ conditions correspond to continuity of excess pressure (ψ) and normal excess velocity (proportional to $\hat{\mathbf{n}}\cdot\nabla\psi$).

Conditions (4)–(7) are illustrative. If we indicate the solutions for two arbitrary directions of incidence $\hat{\mathbf{r}}_1$, $\hat{\mathbf{r}}_2$ by ψ_1, ψ_2 (each subject to the same conditions on $\mathcal{S}$ and $\mathcal{V}$), then the subsequent development applies for general obstacles for which Green's surface integral for ψ_1, ψ_2 vanishes on $\mathcal{S}$. Introducing a symbolic form (and normalization constants for subsequent applications),

$$(8) \qquad \{v, u\} \equiv c\int [v(\mathbf{r}')\partial_n u(\mathbf{r}') - u\,\partial_n v]\,d\mathcal{S}(\mathbf{r}'), \qquad c = \frac{k}{i4\pi}, \frac{1}{i4}, \frac{1}{i2k},$$

the development applies for all cases such that [7], [8]

$$(9) \qquad \{\psi_1, \psi_2\} = 0.$$

By inspection, it is clear that (9) covers the special cases (4)–(6). To show that (7) is also covered, we use (7$\mathcal{S}$) and then apply Gauss' theorem to obtain an integral over $\mathcal{V}$ which vanishes by (7$\mathcal{V}$).

The conditions on the nonvanishing imaginary parts of Z, C, B correspond to energy absorption by the obstacles (lossy obstacles). Changing the signs of the imaginary parts corresponds to energy production (gainy obstacles, or negative-lossy obstacles). If ψ satisfies (4) or (5), or (6) for real Z, or (7) for real B and C, we label it as lossless. More generally, a lossless obstacle subject to (9) also satisfies [7], [8]

$$(10) \qquad \{\psi_1^*, \psi_2\} = 0,$$

where ψ^* is the complex conjugate of ψ. The statement after (9) applies equally for (10).

In general, (10) is not satisfied. If we normalize the incident energy flux vector as

$$(11) \qquad \operatorname{Re}(\phi^*\nabla\phi/ik) = \hat{\mathbf{k}},$$

with unit energy density in unit area normal to $\hat{\mathbf{k}}$, then for lossy obstacles the net energy flow into $\mathcal{S}$ (the absorbed energy) is given by

$$(12) \qquad -\int \operatorname{Re}(\psi^*\nabla\psi/ik)\cdot\hat{\mathbf{n}}\,d\mathcal{S} = -\tfrac{1}{2}\sigma_0\{\psi^*, \psi\} \equiv \sigma_A, \qquad \sigma_0 = 1/ick,$$

where $\sigma_A > 0$ is the absorption cross section. Were $\sigma_A < 0$, we would relabel it $-\sigma_P$ with $\sigma_P > 0$, and cancel the minus signs to discuss energy production (negative-absorption).

The two classes of problems we consider are determined by different conditions for $\psi - \phi$ on the surface

$$S = \lim_{r\to\infty} S(r). \tag{13}$$

We label S as the spherical surface at infinity, the volume V external to $\mathcal{S}$ as the shell volume, and suppress limit operations in general.

One problem, the scattering or radiating obstacle, is well-posed if we supplement (1) and (9) by the Magnus–Wilcox form of Sommerfeld's radiation condition [3], [4] for $\psi - \phi = u(\mathbf{r})$,

$$I = \int |\partial_r u - iku|^2 \, dS = 0. \tag{14}$$

This condition leads to [4]

$$\int |u|^2 \, dS \equiv b, \qquad \int |\partial_r u|^2 \, dS \equiv b' \tag{15}$$

with b and b' finite, and to [4]

$$\int \operatorname{Re}(u^* \partial_r u/ik) \, dS = \int \operatorname{Re}(u^* \nabla u/ik) \cdot \hat{\mathbf{n}} \, d\mathcal{S} = \tfrac{1}{2}\sigma_0\{u^*, u\} \equiv \sigma_S > 0, \tag{16}$$

where σ_S is the scattering cross section. Equation (16) states there is net energy outflow from the obstacle [3], [4]. The associated wave $ue^{-i\omega t}$ will be shown to have phase $kr - \omega t$ for $r \sim \infty$ corresponding to a wave $u^+ e^{-i\omega t}$ outgoing from $r = 0$. The sum $\sigma = \sigma_A + \sigma_S$, the total energy cross section, will be shown proportional to the interference terms of ϕ and u in the forward direction $\hat{\mathbf{r}} = \hat{\mathbf{k}}$. Thus σ is a measure of the energy the obstacle derives by interference of ϕ and u and dissipates via absorption (conversion to heat, to radiation with different ω, etc.) or diverts as scattered radiation over the continuum of directions $\hat{\mathbf{r}}$.

Because $\sigma_S > 0$ in (16), the energy is outflowing from the obstacle which thus represents a secondary source of radiation. Were we to change the sign in (14), we would obtain $\sigma_S < 0$, and energy would flow into the obstacle corresponding to a sink of radiation. The associated wave $u^- e^{-\omega t}$ would have phase $-kr - \omega t$ for $r \sim \infty$ corresponding to a wave incoming to $r = 0$.

The second problem we consider, for $\psi = \psi' = \phi + u'$ subject to (1) and (9), is that of the radiationless obstacle

$$\int \operatorname{Re}(u'^* \partial_r u'/ik) \, dS \equiv 0, \tag{17}$$

$$u' = \tfrac{1}{2}(u^+ - u^-), \qquad I^\pm = \int |\partial_r u^\pm \mp iku^\pm|^2 \, dS = 0, \tag{18}$$

where u' represents a standing wave, an appropriate combination of an outgoing and incoming wave corresponding to zero net radiation. The S integral in (18) gives (15) for $u^\pm$, and $\sigma_S^\pm \gtrless 0$ for (16). In order to facilitate discussion of u', we first consider the radiating obstacle specified by u subject to (14). The development for u covers aspects common to both u and u' with emphasis on procedures that apply for both, but some elementary properties of u and its associated scattering amplitude g are included to delineate differences between u and u'.

2. The scattering obstacle; the amplitude g. We apply Green's theorem to u and

$$\mathscr{H}_0(k|\mathbf{r}-\mathbf{r}'|) = \mathscr{H}_0(\rho) = h_0^{(1)}(\rho), H_0^{(1)}(\rho), e^{i\rho}, \tag{19}$$

where $\mathscr{H}_0$, the appropriate zeroth order Hankel function of the first kind, is the normalized free-space Green's function. We consider a fixed observation point $\mathbf{r}$ in the shell $V(\mathbf{r}')$, exclude the singularity of $\mathscr{H}_0$ by a limiting sphere with radius $|\mathbf{r}-\mathbf{r}'|\to 0$, and obtain u as the difference of Green's surface integrals over $\mathscr{S}$ and S

$$u(\mathbf{r}) = \{\mathscr{H}_0(k|\mathbf{r}-\mathbf{r}'|), u(\mathbf{r}')\} - c\int (\mathscr{H}_0\partial_n u - u\,\partial_n\mathscr{H}_0)\,dS. \tag{20}$$

We proceed essentially as in [4].

The integral over S can be written in terms of

$$I_1 = \int \mathscr{H}_0(\partial_n u - iku)\,dS, \quad I_2 = \int \mathscr{H}_0(u/r')\,dS, \quad \mathscr{H}_0 \sim \mathscr{H}(kr')\,e^{-ik\mathbf{r}\cdot\hat{\mathbf{r}}'},$$

where $|\mathscr{H}| \propto (r')^{-(m-1)/2}$ and $dS \propto (r')^{m-1}$ for $m = 3, 2, 1$; I_2 does not arise for the one-dimensional case, and the limit for $r'\to\infty$ is implicit. Using Schwarz's inequality, and comparing $|I_1|^2$ with I of (14) and $|I_2|^2$ with b of (15), we obtain $\int(\cdot)\,dS = 0$ in (20). Thus

$$u(\mathbf{r}) = \{\mathscr{H}_0(k|\mathbf{r}-\mathbf{r}'|), u(\mathbf{r}')\} = \{\mathscr{H}_0, u\}, \tag{21}$$

with the brace operation over $\mathscr{S}(\mathbf{r}')$ as in (8) or over any surface that separates $\mathscr{S}$ from $\mathbf{r}$, is a radiative solution of Helmholtz's equation [3], [4]. For $r\sim\infty$,

$$u(\mathbf{r}) \sim \mathscr{H}(kr)g(\hat{\mathbf{r}}, \hat{\mathbf{k}}), \tag{22}$$

$$g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) \equiv \{e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}, u(\mathbf{r}')\} = \{\phi_{-\hat{\mathbf{r}}}, u\} = \{\phi_{\hat{\mathbf{r}}}^*, u\}, \tag{23}$$

with $\mathscr{H}$ as the asymptotic form of $\mathscr{H}_0$, and $g(\hat{\mathbf{r}}, \hat{\mathbf{k}})$ as the normalized scattering amplitude [7], [8]. The function $\mathscr{H}$ differs from $\mathscr{H}_0$ only in two dimensions, i.e.,

$$\mathscr{H}(\rho) = \frac{e^{i\rho}}{i\rho}, \left(\frac{2}{\pi\rho}\right)^{1/2} e^{i\rho - i\pi/4}, e^{i\rho}. \tag{24}$$

In one-dimensional problems we use $\rho/k = |z| = \pm z$ and $g(\pm\hat{\mathbf{z}}, \hat{\mathbf{z}})$. For all cases we write the phase of the associated wave as $kr-\omega t$ corresponding to a wave outgoing from $r = 0$.

The scattering amplitude g not only determines u for $r\sim\infty$ (i.e., in the far-field $kr\gg 1$, $r\gg a$) but for all $r>a$, and at least for some $r\leqq a$. Substituting Noether's [6] and Sommerfeld's [5] complex spectral representation for $\mathscr{H}_0$ into (21) and using the definition (23) for g, we obtain [8], [7] a corresponding spectral form for u. Thus, at least for $r>a$ for all $\hat{\mathbf{r}}$, and for r greater than the obstacle's projection on $\hat{\mathbf{r}}$,

$$u(\mathbf{r}) = \int_c e^{ik\hat{\mathbf{r}}_c\cdot\mathbf{r}} g(\hat{\mathbf{r}}_c, \hat{\mathbf{k}}). \tag{25}$$

In three dimensions [8], $\hat{\mathbf{r}}_c = \hat{\mathbf{r}}(\theta_c, \varphi_c)$ and $\int_c = (1/2\pi)\int d\Omega(\theta_c, \varphi_c)$ with contours [6] as for $h_0^{(1)}$; in two [7], $\hat{\mathbf{r}}_c = \hat{\mathbf{r}}(\theta_c)$ and $\int_c = (1/\pi)\int d\theta_c$ with contour [5] as for $H_0^{(1)}$; in one, $\hat{\mathbf{r}}_c = \pm\hat{\mathbf{z}}$ and $\int_c$ selects the sign corresponding to $z = \pm|z|$. (The associated integral operator for $\mathscr{J}_0 = \operatorname{Re}\mathscr{H}_0$ is the mean ($\mathscr{M}$) over real directions of observation.) The brace operator in (23) specifies g in terms of u, and the $\int_c$ operator in (25) is an inverse for u in terms of g. Other inverses, and complete convergent and asymptotic expansions of u for large kr (in terms of g and its derivatives with respect to angles) are discussed in the literature [7]–[10].

From the general obstacle condition (9), in terms of $\psi_i = \phi_i + u_i$, we have

$$\{\psi_1, \psi_2\} = \{\phi_1 + u_1, \phi_2 + u_2\} = \{\phi_1, u_2\} + \{u_1, \phi_2\} = 0, \tag{26}$$

where $\{\phi_1, \phi_2\}$ vanished by Green's theorem and $(\nabla^2 + k^2)\phi_i = 0$ in $\mathcal{V}$. The term $\{u_1, u_2\}$ vanished by using Green's theorem to convert the $\mathcal{S}$ integral to V plus S integrals; the V integral is zero from (1) for u, and the S integral is zero because the asymptotic form u of (22) is appropriate. From (26), we have $\{\phi_1, u_2\} = \{\phi_2, u_1\}$; by the definition of g as in (23) this corresponds to

$$g(-\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) = g(-\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1), \qquad g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = g(-\hat{\mathbf{k}}, -\hat{\mathbf{r}}), \tag{27}$$

i.e., to the usual reciprocity theorem.

Proceeding similarly for $\{\psi_1^*, \psi_2\}$, we have

$$\begin{aligned} \{\psi_1^*, \psi_2\} &= \{\phi_1^*, u_2\} + \{u_1^*, \phi_2\} + \{u_1^*, u_2\} \\ &= g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) + g^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) + c \int (u_1^* \partial_n u_2 - u_2 \partial_n u_1^*)\, dS, \end{aligned} \tag{28}$$

where we used $\{u_1^*, \phi_2\} = -\{\phi_2, u_1^*\} = \{\phi_2^*, u_1\}^*$. In the integral over S, we substitute the asymptotic u of (22) to obtain

$$\{\psi_1^*, \psi_2\} = g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) + g^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) + 2\mathcal{M} g^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2), \tag{29}$$

where $\mathcal{M}$ is the mean over all values of $\hat{\mathbf{r}}$. Explicitly

$$\mathcal{M}_3 = \frac{1}{4\pi} \int d\Omega(\theta, \varphi) = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta, \qquad \mathcal{M}_2 = \frac{1}{2\pi} \int_0^{2\pi} d\theta, \tag{30}$$

and $\mathcal{M}_1$ is one-half the forward and back values.

For lossless obstacles, (10) requires $\{\psi_1^*, \psi_2\} = 0$ and (29) reduces to

$$-g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) - g^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) = 2\mathcal{M} g^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2). \tag{31}$$

In particular, if the obstacle has inversion symmetry (so that $\mathbf{r}$ on $\mathcal{S}$ implies $-\mathbf{r}$ on $\mathcal{S}$), then from (27) we have $g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) = g(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1)$, and from (31),

$$-\mathrm{Re}\, g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) = \mathcal{M} g^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2); \tag{32}$$

for such symmetry, the right side is real. If $|\mathrm{Im}\, g| \gg |\mathrm{Re}\, g|$, then $\mathrm{Re}\, g$ is of order $(\mathrm{Im}\, g)^2$ and (32) may be used[11,12] to construct $\mathrm{Re}\, g$ from approximations of $\mathrm{Im}\, g$.

From (31) for $\hat{\mathbf{r}}_1 = \hat{\mathbf{r}}_2 = \hat{\mathbf{k}}$, we obtain the forward scattering theorem with no requirement of symmetry,

$$-\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}}) = \mathcal{M} |g(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2 = \tfrac{1}{2}\{u^*, u\} = \sigma_S/\sigma_0 \tag{33}$$

in terms of σ_S of (16). This relation corresponds to the energy theorem for lossless scatterers: the energy derived by the obstacle via interference of the incident wave ϕ with the scattered wave u in the forward direction $\hat{\mathbf{k}}$ is radiated over all directions $\hat{\mathbf{r}}$. The theorem also states that if $\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}}) = 0$ (if there is no interference of ϕ and u), then $g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = 0$ for all $\hat{\mathbf{r}}$ (then there is no scattering): $\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}})$ vanishes only if $\mathcal{M}|g(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2$ vanishes, and this requires $\mathrm{Re}^2 g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) + \mathrm{Im}^2 g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = 0$, from which $\mathrm{Re}\, g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = \mathrm{Im}\, g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = 0$. Since $\hat{\mathbf{k}}$ is arbitrary, $\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}}) = 0$ implies $g(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = 0$ for all $\hat{\mathbf{r}}$.

For lossy obstacles, from (29) in terms of (12) and (16),

$$(34)\qquad \begin{aligned} -\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}}) &= -\tfrac{1}{2}\{\psi^*, \psi\} + \tfrac{1}{2}\{u^*, u\} = (\sigma_A + \sigma_S)/\sigma_0, \\ \sigma_0 &= \frac{4\pi}{k^2}, \frac{4}{k}, 2, \end{aligned}$$

which states that the energy derived by the obstacle via interference of ϕ with u is balanced by the energy absorbed and reradiated. Because σ_A and σ_S are positive, both must vanish in order for $\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}})$ to vanish; if there is no interference, there is neither absorption nor scattering (no obstacle). On the other hand for a gainy obstacle, such that excitation by ϕ causes it to produce energy $-\sigma_A = \sigma_P > 0$, then

$$(35)\qquad -\sigma_0 \,\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}}) = -\sigma_P + \sigma_S$$

may vanish: if $\sigma_P = \sigma_S$, no energy is contributed via interference of ϕ and u; the energy σ_P produced by the obstacle balances the energy it radiates. Although $\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}})$ may vanish for $\sigma_S \neq 0$ in (35), the converse is ruled out (except for the trivial case); the vanishing of $\sigma_S = \sigma_0 \mathcal{M} |g(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2$ implies $\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}}) = 0$, so that σ_P would also vanish (no obstacle). If $\sigma_P > \sigma_S$, then $\mathrm{Re}\, g(\hat{\mathbf{k}}, \hat{\mathbf{k}}) > 0$ and the constructive interference between ϕ and u is a result of the energy produced by the obstacle.

For the general case specified by (28), the analogue of (34) is

$$(36)\qquad \begin{aligned} -g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) - g^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) &= -\{\psi_1^*, \psi_2\} + \{u_1^*, u_2\} = 2(\sigma_A^{2,1} + \sigma_S^{2,1})/\sigma_0, \\ \sigma_S^{2,1} &= \sigma_0 \mathcal{M} g^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2). \end{aligned}$$

If the obstacle has inversion symmetry, the left side reduces to $-2\,\mathrm{Re}\, g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2)$ as in (32), and $\sigma_A^{2,1}$ and $\sigma_S^{2,1}$ are real.

3. The nonscattering obstacle; the amplitude g'. We consider the problem for $\psi' = \phi + u'$ defined by (1), (2), (9), (17) and (18). The problem for ψ' differs from the scattering problem for ψ in that the radiation condition (14) designed to correspond to net outgoing radiation (16) is replaced by the nonradiating (17) standing wave (18) conditions. Relabeling the outgoing Green's function (19) and introducing its complex conjugates as the incoming one,

$$(37)\qquad \mathcal{H}_0 = \mathcal{H}_0^+ = \mathcal{H}_0^{(1)}, \qquad \mathcal{H}_0^* = \mathcal{H}_0^- = \mathcal{H}_0^{(2)}$$

we see that the standing wave Green's function

$$(38)\qquad \tfrac{1}{2}(\mathcal{H}_0 - \mathcal{H}_0^*) = i\,\mathrm{Im}\,\mathcal{H}_0 = i\mathcal{N}_0$$

has the same singularity as $\mathcal{H}_0$.

Applying Green's theorem to $i\mathcal{N}_0$ and u', and proceeding as for (20), we obtain

$$(39)\qquad \begin{aligned} u'(\mathbf{r}) &= \{i\mathcal{N}_0(k|\mathbf{r}-\mathbf{r}'|), u'(\mathbf{r}')\} - (I_{11} + I_{22} - I_{12} - I_{21}), \\ u' &= \tfrac{1}{2}(u^+ - u^-) = \tfrac{1}{2}(u^{(1)} - u^{(2)}), \\ 4I_{ij} &= c \int (\mathcal{H}_0^{(i)} \partial_n u^{(j)} - u^{(j)} \partial_n \mathcal{H}_0^{(i)})\, dS. \end{aligned}$$

The integrals I_{11} and I_{22} can be expressed in terms of

$$I_1^{\pm} = \int \mathcal{H}_0^{\pm} (\partial_n u^{\pm} \mp iku^{\pm})\, dS, \qquad I_2^{\pm} = \int \mathcal{H}_0^{\pm} (u^{\pm}/r')\, dS;$$

comparing $|I_1^{\pm}|^2$ with $I^{\pm}$ of (18), and $|I_2^{\pm}|^2$ and b of (15), shows that I_{11} and I_{22} vanish essentially as discussed for (20). The integrals I_{12} and I_{21} involve analogues of I_2 in terms of $\mathscr{H}^{\pm}u^{\mp}$ which also vanish by comparison with b, so that the key integrals are

$$\int \mathscr{H}_0^{\mp}(\partial_n u^{\pm} \pm iku^{\pm})\,dS = J^{\pm} \pm i2k\int \mathscr{H}_0^{\mp}u^{\pm}\,dS,$$

where $J^{\pm}$, differing from $I_1^{\pm}$ only in the interchange of $\mathscr{H}^{+}$ and $\mathscr{H}^{-}$, vanishes by (18). Consequently, to eliminate $\int(\cdot)\,dS$ of (39) we require

$$R = \int (\mathscr{H}_0^{-}u^{+} - \mathscr{H}_0^{+}u^{-})\,dS = 0 \tag{40}$$

which constrains the amplitudes of u^{+} and u^{-}. Thus subject to (40), we have

$$u'(\mathbf{r}) = \{i\mathcal{N}_0(k|\mathbf{r}-\mathbf{r}'|), u'(\mathbf{r}')\} = \tfrac{1}{2}\{\mathscr{H}_0^{+}, u'\} - \tfrac{1}{2}\{\mathscr{H}_0^{-}, u'\} \tag{41}$$

For $r \sim \infty$,

$$u'(\mathbf{r}) \sim \tfrac{1}{2}\mathscr{H}^{+}(kr)g'(\hat{\mathbf{r}}, \hat{\mathbf{k}}) - \tfrac{1}{2}\mathscr{H}^{-}(kr)g'(-\hat{\mathbf{r}}, \hat{\mathbf{k}}), \tag{42}$$

$$g'(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = \{e^{-ik\hat{\mathbf{r}}\cdot\mathbf{r}'}, u'\} \tag{43}$$

where $\mathscr{H}^{+} = \mathscr{H}$ of (24), $\mathscr{H}^{-} = \mathscr{H}^{*}$, and g' is the amplitude for the radiationless problem. For all cases, $u'e^{-i\omega t}$ consists of outgoing plus incoming waves with corresponding phases $kr - \omega t$ and $-kr - \omega t$.

To ascertain the import of (40) we consider $u^{\pm}$ further. From $I_{11} = I_{22} = 0$, and from the signs associated with $\mathscr{H}_0^{\pm}(\rho)/i$ for $\rho \to 0$, we have

$$u^{\pm}(\mathbf{r}) = \pm\{\mathscr{H}_0^{\pm}, u^{\pm}\}, \qquad u^{\pm} \sim \pm\mathscr{H}^{\pm}\{e^{\mp ik\hat{\mathbf{r}}\cdot\mathbf{r}'}, u^{\pm}\} = \pm\mathscr{H}^{\pm}g^{\pm}(\pm\hat{\mathbf{r}}), \tag{44}$$

and consequently

$$u'(\mathbf{r}) = \tfrac{1}{2}(u^{+} - u^{-}) \sim \tfrac{1}{2}\mathscr{H}^{+}g^{+}(\hat{\mathbf{r}}) + \tfrac{1}{2}\mathscr{H}^{-}g^{-}(-\hat{\mathbf{r}}), \tag{45}$$

where $\hat{\mathbf{k}}$ was suppressed as unessential. Alternatively, from (41) with u' in the braces replaced by $\frac{1}{2}(\mathrm{u}^{+} - \mathrm{u}^{-})$,

$$\begin{aligned} u' &= \tfrac{1}{4}\{\mathscr{H}_0^{+}, u^{+} - u^{-}\} - \tfrac{1}{4}\{\mathscr{H}_0^{-}, u^{+} - u^{-}\}, \\ u' &\sim \tfrac{1}{4}\mathscr{H}^{+}[g^{+}(\hat{\mathbf{r}}) - g^{-}(\hat{\mathbf{r}})] - \tfrac{1}{4}\mathscr{H}^{-}[g^{+}(-\hat{\mathbf{r}}) - g^{-}(-\hat{\mathbf{r}})]. \end{aligned} \tag{46}$$

From (45) and (46),

$$g^{+}(\hat{\mathbf{r}}) = \tfrac{1}{2}[g^{+}(\hat{\mathbf{r}}) - g^{-}(\hat{\mathbf{r}})], \qquad -g^{-}(-\hat{\mathbf{r}}) = \tfrac{1}{2}[g^{+}(-\hat{\mathbf{r}}) - g^{-}(-\hat{\mathbf{r}})], \tag{47}$$

so that the amplitudes are related by

$$g^{+}(\hat{\mathbf{r}}) = -g^{-}(\hat{\mathbf{r}}) \tag{48}$$

for all $\hat{\mathbf{r}}$. From (42), (45) and (48), we also have

$$g'(\hat{\mathbf{r}}, \hat{\mathbf{k}}) = g^{+}(\hat{\mathbf{r}}), \qquad g'(-\hat{\mathbf{r}}, \hat{\mathbf{k}}) = -g^{-}(-\hat{\mathbf{r}}) = g^{+}(-\hat{\mathbf{r}}) \tag{49}$$

Substituting (44) into the constraint $R = 0$ of (40), we see that it corresponds to

$$R = \int \mathscr{H}^{-}\mathscr{H}^{+}[\phi_{\hat{\mathbf{r}}}g^{+}(\hat{\mathbf{r}}) + \phi_{-\hat{\mathbf{r}}}g^{-}(-\hat{\mathbf{r}})]\,dS = 0, \tag{50}$$

or equivalently, to the requirement that

$$\mathcal{M}[\phi_{\hat{\mathbf{r}}}g'(\hat{\mathbf{r}}, \hat{\mathbf{k}}) - \phi_{-\hat{\mathbf{r}}}g'(-\hat{\mathbf{r}}, \hat{\mathbf{k}})] = 0. \tag{51}$$

That the mean value in (51) vanishes follows from the general form

$$\mathcal{M}[f(\hat{\mathbf{r}})-f(-\hat{\mathbf{r}})]=0, \tag{52}$$

i.e., $\hat{\mathbf{r}}$ is a dummy and the operator $\mathcal{M}$ means integrate over all $\hat{\mathbf{r}}$. Thus the constraint $R=0$ is satisfied and the procedure is fully consistent.

We now apply (41)–(43) to parallel the development (26)–(36). Because $\psi'=\phi+u'$ satisfies the general obstacle condition (9), we have

$$\{\psi_1', \psi_2'\}=\{\phi_1, u_2'\}+\{u_1', \phi_2\}+\{u_1', u_2'\}_S=0, \tag{53}$$

where $\{\phi_1, \phi_2\}$ vanished, and $\{u_1', u_2'\}$ was converted to an S integral via the discussion for (26). On S, we use (42) to reduce the integral to

$$\mathcal{M}[g'(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1)g'(-\hat{\mathbf{r}}, \hat{\mathbf{r}}_2)-g'(-\hat{\mathbf{r}}, \hat{\mathbf{r}}_1)g'(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2)]=0, \tag{54}$$

which vanishes by (52). Thus (53) reduces to $\{\phi_1, u_2'\}+\{u_1', \phi_2\}=0$, and proceeding as for (27) in terms of (43), we obtain

$$g'(-\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2)=g'(-\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1), \qquad g'(\hat{\mathbf{r}}, \hat{\mathbf{k}})=g'(-\hat{\mathbf{k}}, -\hat{\mathbf{r}}), \tag{55}$$

i.e., the same reciprocity theorem as for g.

Proceeding similarly for $\{\psi_1'^*, \psi_2'\}$, we construct the analogue of (28),

$$\{\psi_1'^*, \psi_2'\}=g'(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2)+g'^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1)+\{u_1'^*, u_2'\}_S. \tag{56}$$

Using the asymptotic u' of (42) on S,

$$\{u_1'^*, u_2'\}_S=\tfrac{1}{2}\mathcal{M}[g'^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1)g'(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2)-g'^*(-\hat{\mathbf{r}}, \hat{\mathbf{r}}_1)g'(-\hat{\mathbf{r}}, \hat{\mathbf{r}}_2)]=0, \tag{57}$$

which vanishes by (52). The special case of (57) for $\hat{\mathbf{r}}_1=\hat{\mathbf{r}}_2=\hat{\mathbf{k}}$ corresponds to the analogue of the scattering cross section:

$$\sigma_S'=\sigma_0\tfrac{1}{2}\mathcal{M}[|g'(\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2-|g'(-\hat{\mathbf{r}}, \hat{\mathbf{k}})|^2]=0. \tag{58}$$

Thus, the obstacle is radiationless in the sense required: there is no net outflow or inflow of energy in the form of radiation over the continuum of directions $\hat{\mathbf{r}}$. Consequently (56) reduces to

$$-g'(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2)-g'^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1)=-\{\psi_1'^*, \psi_2'\}=2\sigma_A'(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1)/\sigma_0, \tag{59}$$

i.e., to the analogue of (36) for no scattering losses. In particular, for $\hat{\mathbf{r}}_1=\hat{\mathbf{r}}_2=\hat{\mathbf{k}}$,

$$-\mathrm{Re}\, g'(\hat{\mathbf{k}}, \hat{\mathbf{k}})=-\tfrac{1}{2}\{\psi'^*, \psi'\}=\sigma_A'/\sigma_0, \tag{60}$$

where σ_A' is the absorption cross section of the radiationless obstacle. The energy theorem (60) states that the energy the obstacle derives via interference of ϕ and u' in the forward direction is dissipated by absorption. The presence of the interference term qualifies the label radiationless introduced for brevity: the label radiationless is defined by (17) and (58).

If the obstacle is lossless, then (59) reduces to

$$g'(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2)+g'^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1)=0. \tag{61}$$

If the obstacle has inversion symmetry,

$$\mathrm{Re}\, g'(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2)=0 \tag{62}$$

and even if it doesn't, the forward value of (61) satisfies

$$\mathrm{Re}\, g'(\hat{\mathbf{k}}, \hat{\mathbf{k}})=0. \tag{63}$$

There is no interference of ϕ and u' for this case; the obstacle neither scatters nor absorbs, so no energy is called for.

4. Interrelations of g and g'. The *general obstacle condition* defined in (9) was applied as an operator to two solutions ψ_1, ψ_2 for the scattering obstacle in (26) and to two solutions ψ'_1, ψ'_2 for the nonscattering obstacle in (53). This led to the same reciprocity relation, (27) for g and (55) for g', for the values of the amplitude with arguments involving two arbitrary directions of incidence $\hat{\mathbf{r}}_1$, $\hat{\mathbf{r}}_2$. We now apply the operator to ψ_1 and ψ'_2, where ψ and ψ' satisfy the same conditions on $\mathscr{S}$ and $\mathscr{V}$. This leads to an integral equation for g in terms of g', as well as to the inverse.

As discussed for (26) and (53),

$$\{\psi_1, \psi'_2\} = \{\phi_1 + u_1, \phi_2 + u'_2\} = \{\phi_1, u'_2\} - \{\phi_2, u_1\} + \{u_1, u'_2\}_S = 0. \tag{64}$$

Using the definitions of g and g' as in (23) and (43), and the asymptotic u and u' as in (22) and (42), we obtain

$$\begin{aligned} g(-\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) &= g'(-\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) + \mathscr{M} g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g'(-\hat{\mathbf{r}}, \hat{\mathbf{r}}_2) \\ &= g'(-\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) + \mathscr{M} g'(-\hat{\mathbf{r}}_2, \hat{\mathbf{r}}) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1), \end{aligned} \tag{65}$$

where the second version follows from the reciprocity theorem for g'. Relabeling $-\hat{\mathbf{r}}_2$ as $\hat{\mathbf{r}}_2$, we have

$$g(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) = g'(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) + \mathscr{M} g'(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) \tag{66}$$

which applies as an integral equation for g in terms of g', or for the converse. This form and a related one arose earlier [1] in reducing functional equations for multiple-scattering problems which involved an operator consisting of $\mathscr{M}$ plus additional operations. The related form follows from (66) by using (52) as $\mathscr{M} f(\hat{\mathbf{r}}) = \mathscr{M} f(-\hat{\mathbf{r}})$ and then applying the reciprocity theorems. We construct $\mathscr{M} g'(\hat{\mathbf{r}}_2, -\hat{\mathbf{r}}) g(-\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) = \mathscr{M} g'(\hat{\mathbf{r}}, -\hat{\mathbf{r}}_2) g(-\hat{\mathbf{r}}_1, \hat{\mathbf{r}}) = g(-\hat{\mathbf{r}}_1, -\hat{\mathbf{r}}_2) - g'(-\hat{\mathbf{r}}_1, -\hat{\mathbf{r}}_2)$, and then relabel $-\hat{\mathbf{r}}_i$ as $\hat{\mathbf{r}}_i$ to obtain

$$g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) = g'(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) + \mathscr{M} g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}) g'(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2). \tag{67}$$

Since $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$ are arbitrary, the only substantive difference in (66) and (67) is in the location of $\hat{\mathbf{r}}$. Form (67) follows trivially from (66) if the obstacle has inversion symmetry, but the present procedure shows it holds for all shapes.

Similarly we apply the generalized energy-type operation considered in (28) for ψ and (56) for ψ' to the mixed set ψ'^*_1, ψ_2 to obtain

$$\begin{aligned} \{\psi'^*_1, \psi_2\} &= \{\phi^*_1, u_2\} + \{u'^*_1, \phi_2\} + \{u'^*_1, u_2\}_S \\ &= g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) + g'^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) + \mathscr{M} g'^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2). \end{aligned} \tag{68}$$

For lossless obstacles,

$$g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) = -g'^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) - \mathscr{M} g'^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2), \tag{69}$$

as follows directly from (66) by using (61) for lossless g'. From (69) and (61) (or from (66), (67), and (61)), we have

$$\begin{aligned} -g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) - g^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) &= \mathscr{M}[g'^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2) + g'(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2) g^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1)] \\ &= \mathscr{M}[g'^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2) + g'(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}) g^*(\hat{\mathbf{r}}_2, \hat{\mathbf{r}})]. \end{aligned} \tag{70}$$

If the obstacle has inversion symmetry, then

$$-\mathrm{Re}\, g(\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2) = \mathrm{Re}\, \mathscr{M} g'^*(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2) = \mathscr{M}\, \mathrm{Im}\, g'(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1)\, \mathrm{Im}\, g(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2), \tag{71}$$

where $g' = i\, \mathrm{Im}\, g'$.

For nonsymmetrical obstacles, from (66) and (67) in briefer notation

$$g_{12}=g'_{12}+\mathcal{M}g'_{1r}g_{r2}, \qquad g^*_{21}=g'^*_{21}+\mathcal{M}(g_{2r}g'_{r1})^*. \tag{72}$$

Using (66) for lossless obstacles, i.e., $g'^*_{21}=-g'_{12}$, we convert g^*_{21} to

$$g^*_{21}=-g'_{12}-\mathcal{M}g'_{1r}g^*_{2r}, \tag{73}$$

which together with g_{12} of (72) leads to

$$g_{12}+g^*_{21}=\mathcal{M}g'_{1r}(g_{r2}-g^*_{2r}), \qquad g_{12}-g^*_{21}=2g'_{12}+\mathcal{M}g'_{1r}(g_{r2}+g^*_{2r}). \tag{74}$$

Writing

$$\begin{gathered} g_{12}=S_{12}+A_{12}, \quad g_{21}=S_{12}-A_{12}, \quad S_{12}=\tfrac{1}{2}(g_{12}+g_{21}), \quad A_{12}=\tfrac{1}{2}(g_{12}-g_{21}), \\ p_{12}=g_{12}+g^*_{21}=2\operatorname{Re}S_{12}+2\mathrm{i}\operatorname{Im}A_{12}, \quad m_{12}=g_{12}-g^*_{21}=2i\operatorname{Im}S_{12}+2\operatorname{Re}A_{12}, \end{gathered} \tag{75}$$

We recast (74) as

$$m_{12}=2g'_{12}+\mathcal{M}_r g'_{1r}p_{r2}, \qquad p_{12}=\mathcal{M}_r g'_{1r}m_{r2}. \tag{76}$$

Thus, introducing $\mathcal{M}_p$, we uncouple m and p by

$$\begin{gathered} m_{12}=2g'_{12}+\mathcal{M}_r g'_{1r}\mathcal{M}_\rho g'_{r\rho}m_{\rho 2}, \\ p_{12}=2\mathcal{M}_r g'_{1r}g'_{r2}+\mathcal{M}_r g'_{1r}\mathcal{M}_\rho g'_{r\rho}p_{\rho 2}, \end{gathered} \tag{77}$$

which supplement the original integral equation (66) when the obstacle is lossless.

5. Supplementary considerations and applications. The form (67) arose earlier for the two-dimensional problem of scattering by a grating of equal spaced parallel cylinders [1]. Using a symbolic procedure and the energy theorem (32) for lossless g having inversion symmetry, it was shown that $\operatorname{Re} g'_{12}=0$ of (62) was satisfied. Writing (67) for the two-dimensional case (with $\hat{\mathbf{r}}_1$, $\hat{\mathbf{r}}_2$ replaced by θ_1, θ_2 with no function-relevant connotation) as

$$\begin{aligned} g(\theta_1,\theta_2)&=g'(\theta_1,\theta_2)+\frac{1}{2\pi}\int_0^{2\pi} g(\theta_1,\theta)g'(\theta,\theta_2)\,d\theta \\ &=\frac{1}{2\pi}\int d\theta[2\pi\delta(\theta-\theta_1)+g(\theta_1,\theta)]g'(\theta,\theta_2) \end{aligned} \tag{78}$$

provides the starting point [1, (55)] of the earlier development. The corresponding symbolic version in [1, (55a)],

$$Pg'=g, \qquad g'=P^{-1}g, \tag{79}$$

was used to construct [1, (57)]

$$g'=P^{-1}g=|P^{-1}|^2P^*g=|P^{-1}|^2\left[g(\theta,\theta_2)+\frac{1}{2\pi}\int g^*(\theta,\theta')g(\theta',\theta_2)\,d\theta'\right] \tag{80}$$

Applying the two-dimensional case case of (32) led to

$$\operatorname{Re} g'(\theta_1,\theta_2)=0, \qquad g'(\theta_1,\theta_2)=|P^{-1}|^2 i\operatorname{Im} g(\theta,\theta_2) \tag{81}$$

where the first equality corresponds to (62); for the second, we invert via $P^*(Pg')=P^*g$ to reconstruct (32). See [1] for applications of (78) and (81) in reducing the multiple scattering amplitude.

The integral equation (78) was also applied in [1, (104)] to circular cylinders with amplitudes

$$(82)\qquad g(\theta,\alpha)=\sum_{n=-\infty}^{\infty} a_n e^{in(\theta-\alpha)},\qquad g'(\theta,\alpha)=\sum a'_n e^{in(\theta-\alpha)}$$

to obtain

$$(83)\qquad a'_n=a_n-a'_na_n,\qquad a'_n=a_n/(1+a_n)$$

in terms of the known scattering coefficients a_n. Using g in the energy theorem (32) for lossless scatterers led to $-\text{Re}\, a_n=|a_n|^2$, and consequently to

$$(84)\qquad a'_n=i\,\text{Im}\, a_n/|1+a_n|^2$$

as discussed for [1, (104)] and exhibited for the boundary conditions (4) and (5).

The coefficients a_n and a'_n corresponding essentially to all conditions (4)–(7) were considered before [11] in the context of scattering by random distributions of radially symmetric obstacles. Thus, all cases are covered by

$$(85)\qquad a_n=-\frac{R_n\mathscr{J}_n-\mathscr{J}'_n}{R_n\mathscr{H}_n-\mathscr{H}'_n},\quad \mathscr{J}_n=\mathscr{J}_n(ka),\quad \mathscr{H}_n=\mathscr{J}_n+i\mathscr{N}_n=\mathscr{H}^{(1)}_n,$$

where $\mathscr{J}=j$, J and $\mathscr{H}=h$, H, and the prime indicates differentiation with respect to argument. (See [11] for one-dimensional analogues for $\mathscr{J}$, $\mathscr{H}$ for $n=0,1$.) for the boundary conditions (4), (5), (6) we let $R=\infty$, 0, Z/k respectively; for the transition conditions (7),

$$(86)\qquad R_n=\zeta\mathscr{J}'_n(Ka)/\mathscr{J}_n(Ka),\qquad \zeta=B\eta=(CB)^{1/2}$$

where ζ represents an impedance.

From (83), we have

$$(87)\qquad a'_n=i(R_n\mathscr{J}_n-\mathscr{J}'_n)/(R_n\mathscr{N}_n-\mathscr{N}'_n)\equiv i(R_n\mathscr{J}_n-\mathscr{J}'_n)/D_n$$

where $\mathscr{J}_n$ and $\mathscr{N}_n$ are real functions of ka, and R_n is real if the parameters Z or B and K are real (lossless obstacles). In general

$$(88)\qquad \text{Re}\, a'_n=W\,\text{Im}\, R_n/|D_n|^2=-W|\text{Im}\, R_n|/|D_n|^2,\qquad W=1/(ka)^2,\,2/\pi ka,$$

where $W=\mathscr{J}_n\mathscr{N}'_n-\mathscr{J}'_n\mathscr{N}_n$ is the Wronskian. Inverting the development of (84),

$$(89)\qquad a_n=\frac{a'_n}{1-a'_n}=\frac{a'_n-|a'_n|^2}{|1-a'_n|^2},\qquad a_n+|a_n|^2=\frac{a'_n}{|1-a'_n|^2},$$

which for lossless obstacles yields

$$(90)\qquad -\text{Re}\, a_n=\frac{|a'_n|^2}{1+|a'_n|^2}=|a_n|^2.$$

More generally

$$(91)\qquad -\text{Re}\, a_n-|a_n|^2=\frac{-\text{Re}\, a'_n}{|1-a'_n|^2},\qquad |a_n|^2=\frac{|a'_n|^2}{|1-a'_n|^2},$$

with $-\text{Re}\, a'_n$ as in (88) isolates the effects of absorption.

For all such problems we have the forms

$$(92)\qquad a_n=\frac{-\mathbb{J}_n}{\mathbb{H}_n}=\frac{-\mathbb{J}_n}{\mathbb{J}_n+i\mathbb{N}_n}=\frac{a'_n}{1-a'_n},\qquad a'_n=i\frac{\mathbb{J}_n}{\mathbb{N}_n},$$

with $\mathbb{J}_n$ and $\mathbb{N}_n$ real for real parameters. Thus a'_n may be determined by inspection of a_n.

The coefficients for the radially symmetric case give g in the form

$$g(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0)=\sum_{n=0}^{\infty} a_n c_n T_n(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0),\quad c_n=2n+1,\,2-\delta_{n0},\quad T_n=P_n(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0),\,\cos n(\theta-\theta_0). \tag{93}$$

The corresponding representations for ϕ and u are

$$\phi=\sum \mathcal{J}_n(kr)i^n c_n T_n(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0), \tag{94}$$

$$u=\sum \mathcal{H}_n(kr)i^n a_n c_n T_n, \tag{95}$$

and similarly the internal field is the series

$$\psi=\sum \mathcal{J}_n(Kr)i^n b_n c_n T_n. \tag{96}$$

The cases covered by (85) follow from direct application of (4)–(7). We may obtain g of (93) from u of (95) by using $\mathcal{H}_n\sim i^{-n}\mathcal{H}$, and we may also obtain u of (95) by substituting g of (93) into (25).

Similarly, in order to obtain

$$g'(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0)=\sum a'_n c_n T_n(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0) \tag{97}$$

directly instead of via the integral equation $g'[g]$ of (57), we consider the analogous nonscattering problems for (4)–(7) with the outgoing wave form u replaced by the standing wave form

$$u'=\sum i\mathcal{N}_n i^n a'_n c_n T_n,\qquad i\mathcal{N}_n=i\,\mathrm{Im}\,\mathcal{H}_n=\tfrac{1}{2}(\mathcal{H}_n^{(1)}-\mathcal{H}_n^{(2)}), \tag{98}$$

The asymptotic form of u', from $\mathcal{H}_n^{(1)}\sim i^{-n}\mathcal{H}$ and $\mathcal{H}_n^{(2)}\sim i^n\mathcal{H}^*$ is

$$u'\sim\tfrac{1}{2}\mathcal{H}\sum a'_n c_n T_n(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0)-\tfrac{1}{2}\mathcal{H}^*\sum a'_n c_n(-1)^n T_n(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0)=\tfrac{1}{2}\mathcal{H}g'(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0)-\tfrac{1}{2}\mathcal{H}^*g'(-\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}_0), \tag{99}$$

where we use $(-1)^nT_n(x)=T_n(-x)$ to exhibit the required standing wave form (42).

Corresponding to (85) and (87) in terms of (86), the internal coefficients b_n of ψ, and the analogues b'_n of ψ', are given by

$$\begin{aligned} b_n&=-iW/\mathcal{J}_n(Ka)[R_n\mathcal{H}_n-\mathcal{H}'_n]=b'_n/(1-a'_n),\\ b'_n&=-W/\mathcal{J}_n(Ka)[R_n\mathcal{N}_n-\mathcal{N}'_n]=-W/\mathcal{J}_n(Ka)D_n. \end{aligned} \tag{100}$$

Using these and a_n and a'_n, we consider the energy theorems (34) and (60). From $\sigma_A=-\frac{1}{2}\sigma_0\{\psi^*,\psi\}=-\mathrm{Re}\int(\psi^*\partial_r\psi/ik)\,d\mathcal{S}$ we construct

$$\begin{aligned} \sigma_A&=-\mathcal{S}\sum|b_n|^2c_n\,\mathrm{Im}\,[B\eta\mathcal{J}'_n(Ka)\mathcal{J}^*_n(Ka)]\\ &=-\mathcal{S}\sum|b_n|^2c_n|\mathcal{J}_n(Ka)|^2\,\mathrm{Im}\,R_n, \end{aligned} \tag{101}$$

where $\mathcal{S}=4\pi a^2$, $2\pi a$, 2, and $\mathrm{Im}\,R_n\leqq 0$. Similarly σ'_A corresponds to replacing b_n by b'_n. Thus from (100),

$$\sigma'_A=-\mathcal{S}W^2\sum c_n\,\mathrm{Im}\,R_n/|D_n|^2=-\sigma_0\sum\mathrm{Re}\,a'_nc_n=-\sigma_0\,\mathrm{Re}\,g'(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}) \tag{102}$$

with $\mathcal{S}W=\sigma_0$, and $\mathrm{Re}\,a'_n$ as in (88). Similarly,

$$\begin{aligned} \sigma_A&=-\mathcal{S}W^2\sum c_n\,\mathrm{Im}\,R_n/|D_n|^2|1-a'_n|^2=-\sigma_0\sum(\mathrm{Re}\,a_n+|a_n|^2)c_n\\ &=-\sigma_0[\mathrm{Re}\,g(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}})+\mathcal{M}|g(\hat{\mathbf{r}}\cdot\hat{\mathbf{k}})|^2]\\ &=-\sigma_0\,\mathrm{Re}\,g(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}})-\sigma_S. \end{aligned} \tag{103}$$

in terms of $\mathrm{Re}\,a_n$ of (91), and $\sigma_S=\frac{1}{2}\sigma_0\{u^*,u\}=\mathrm{Re}\int(u^*\partial_ru/ik)\,d\mathcal{S}=\sigma_0\mathcal{M}|g(\hat{\mathbf{r}}\cdot\hat{\mathbf{k}})|^2$.

As another application of $g[g']$ of (66), we generalize existing long-wavelength approximations [11], [12] for lossless scatterers to include loss. For lossless ellipsoids, $\operatorname{Im} g_{12}$ to $O(k^5)$ was derived [12] by perturbation procedures, and $\operatorname{Re} g_{12}$ to $O(k^8)$ was obtained from the scattering theorem (32). Thus [12]

$$g(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) = ig^I - g^R, \quad g^I = k^3 A_3 + k^5 A_5 + O(k^7), \quad -g^R = k^6 A_6 + k^8 A_8 + O(k^{10}),$$

$$\begin{aligned} &-A_6(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) = \mathcal{M} A_3(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2) A_3(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1), \\ &-A_8(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) = \mathcal{M}[A_3(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2) A_5(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) + A_5(\hat{\mathbf{r}}, \hat{\mathbf{r}}_2) A_3(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1)], \end{aligned} \tag{104}$$

where the A_n were real functions of real parameters B and $C = B\eta^2$ so that g^I and g^R corresponded to the imaginary and real parts of g. For complex B and C, we write

$$g'(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) \approx ig^I(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) \tag{105}$$

and iterate (66) to obtain

$$g(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) = g'(\hat{\mathbf{r}}_2, \hat{\mathbf{r}}_1) + \mathcal{M} g'(\hat{r}_2, \hat{r}) g'(\hat{\mathbf{r}}, \hat{\mathbf{r}}_1) + O(g'^3), \tag{106}$$

where the term $O(g'^3)$ is $O(k^9)$. Thus, by inspection of the existing results [12] for form (104), we have for complex B and C

$$\operatorname{Im} g(\hat{r}_2, \hat{r}_1) \approx \operatorname{Re} g^I, \qquad \operatorname{Re} g(\hat{r}_2, \hat{r}_1) \approx -\operatorname{Im} g^I - g^R, \tag{107}$$

where $\operatorname{Im} g^I$ and g^R correspond to absorption and scattering losses respectively.

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